Summer Research Paper:

Endogenous Asset Markets and Capital Structure in a General Equilibrium Model*

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Abstract

This paper proposes a general equilibrium model in which two types of agents are present. Managers have exclusive access to a unique production technology for which they seek financing. By proposing financing the manager determines the capital structure of investment. Consumers provide scarce funds in order to buy claims in the risky project and distribute wealth across states. Because of the possibility of default and the scarcity of funds we exhibit an endogenous asset structure. This endogeneity is created through non-feasibility of obtaining proper financing of the managers' project or the unwillingness of the manager to undertake the project.

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1 Introduction

A two period general equilibrium model in fashion of Debreu (1959) and more recently Magill and Quinzii (1996) with uncertainty and two types of agents is introduced. Managers are assumed to have exclusive access to a linear production technology. In order to undertake the investment, and buy the technology, managers must suggest a way to finance the project. Financing is proposed by choosing the capital structure of the project and each manager seeks outside financing due to his poor endowment. If consumers agree to the chosen levels of debt and equity the project is undertaken and the technology is implemented. The managers sell claims of debt and/ or equity to consumers but can also retain some share of equity of their own project. These retained shares are the managers only way of transferring wealth across states. That is, he is prohibited to trade in other securities of other managers. Managers who do not receive financing are deemed to consume their exogenously given endowment.

Consumers are allowed to buy shares of equity and/ or debt of every managers' project. Since the project is risky both claims exhibit risk, however, equity claims enjoy limited liability and debt contracts are collateralized by the residual value of the project.

Two markets are considered, the (real) technology market and the (financial) debt market. Both markets are determined endogenously since accessing the technology and receiving proper financing is not guaranteed for the managers' project. The model therefore offers two possible cases of how a project is not undertaken. First, consumers do not approve the chosen levels of debt and equity by the manager and deny financing the project. Hence, they do not buy (or at least buy not enough) claims in the project. The reasons for this may be that the project is too risky and the manager proposes mainly financing it via equity. Second, managers are simply not willing to undertake the project and therefore block consumers in accessing the technology. Recall that only managers have exclusive access to the initialize the technology. Although artificial, the case might occur if endowments are too large or if projects turn out to be very risky and the probability of getting no payout is high for the manager.

Although only the market for technology and debt are considered the agents are able to create a new asset class, namely equity. This creation is done by holding a proper portfolio of the technology and debt of the managers' project. The payoff structure of both assets is such that it maps the payoff structure of equity.

The paper is structured as follows: The next section shortly reviews the relevant literature. Section 3 introduces the proposed model. Equilibrium for the proposed economy is defined in Section 4. The last section discusses open issues and outlooks for further research with this type of model.

2 Literature Review

The proposed model is in fashion of classical general equilibrium literature which first was comprehensively introduced by Debreu (1959). Extensions including financial markets can be found in Magill and Quinzii (1996). Since collateral is introduced, the model is in spirit of Geanakoplos and Zame (1995) although modeling is different. Hellwig (1981) appears to be the first paper on treating the issues about collateral and default. The paper showed how that the theorem of Modigliani and Miller (1958) survives the possibility of default. Geanakoplos and Zame (1995) and their subsequent revised papers Geanakoplos and Zame (2002) and Geanakoplos and Zame (2007) show that collateral has a profound impact on the prices of commodities and assets, on commodity allocations and the market efficiency. An extension of these models are is for example proposed by Araujo, Pascoa, and Torres-Martinez (2002). They show that collateral avoids Poniz schemes which arise in an infinite horizon incomplete markets economy. Collateral therefore bypasses the artificial debt constraints or transversality conditions imposed by Magill and Quinzii (1994) (and others) which are beyond any budgetary considerations or individual rationality.

Capital structure determination in a general equilibrium model can be found for example in Ammon and Hennessy (2007). They investigate the variation of the capital structure in business cycles using a dynamic general equilibrium model. A simulation study of their model replicates observed macroeconomic data. Capital structure determination truly enjoys most contributions using partial equilibrium models which we do not cite here.

3 The Model

3.1 Time and Uncertainty

We assume two time periods, $t \in \{0, 1\}$. Let the initial state in the first period be denoted by $s = 0$ and let there be a finite set of exogenously realized states, $s \in \{1, ..., S\} = \mathcal{S}$, in the second period. So there are a total of $S + 1$ states in the economy.

3.2 Commodities, Investments and the Debt Market

We consider only one commodity available for consumption in the economy, namely income. Income is made available in each state. Hence, the commodity space is given by $\mathbb{R}^{S+1}$. We omit negative income as consumption and can therefore fix the consumption set equal to the non-negative orthant of the commodity space, i.e., $\mathbb{R}^{S+1}_{+}$.[1]

Regarding the investments suppose that the $k = \{1, ..., K\} = \mathcal{K}$ managers have exclusive access to a unique linear production technology. Each unit of investment in the technology exhibits costs of p_k and returns some positive value ϱ_s^k in states $s \geq 1$. We define the technology market as follows:

$$\mathcal{P} = \begin{pmatrix} -p_1 & \cdots & -p_K \\ \varrho_1^1 & \cdots & \varrho_1^K \\ \vdots & \ddots & \vdots \\ \varrho_S^1 & \cdots & \varrho_S^K \end{pmatrix}. \tag{3.1}$$

Because of the exclusive nature of the production technology the manager is prohibited to trade in other technologies except his own. Consumers enjoy the freedom to trade on the whole market as long as investments are made available to them. In order for the manager to undertake the project a total of i_k^k units of technology investments are needed.

Since the manager is able to finance part of the project via debt we presume q_k to be the price for one unit of debt of firm k. Each debt contract is risky such that managers are not able to return the full amount of debt borrowed from lenders in all states. In states were no default occurs the payoff is one and in states of default the payoff is less than one. Let R_s^k denote the payoff of one unit of debt in states $s \geq 1$ and let the payoff and default mechanism be defined as:

$$R_s^k = \begin{cases} 1, & \text{if } i_k^k \varrho_s^k > d_k^k; \\ \frac{i_k^k \varrho_s^k}{d_k^k}, & \text{otherwise.} \end{cases} \tag{3.2}$$

Default therefore occurs, if the managers are not able to return the total debt amount, d_k^k, to the lenders. That is, if production, $i_k^k \varrho_s^k$, is not sufficient to cover the liabilities. It is clear from (3.2) that in default the payoff of the debt contract is less than one. We define

[1] Recall that, $\mathbb{R}^{S+1}_{+} = \{x \in \mathbb{R}^{S+1} \mid x_s \geq 0 \text{ for } s = 0, 1, ..., S\}$.

the debt market as:

$$\mathcal{D} = \begin{pmatrix} -q_1 & \cdots & -q_K \\ R_1^1 & \cdots & R_1^K \\ \vdots & \ddots & \vdots \\ R_S^1 & \cdots & R_S^K \end{pmatrix} \tag{3.3}$$

And denote the whole market composed of the technology and debt market as $\mathcal{M}$. It will be clear once consumers and managers are defined that agents will create a new asset class considering the market for technology and debt, namely equity.

However, it is already clear that the endogeneity of the technology and debt market arise because of the proposed financing of the manager.

3.3 Managers

The purpose of the manager is to seek the proper financing to undertake the risky project and invest in the technology. Three financing alternatives are made available to the manager. First and second, the manager sells debt and equity contracts to consumers. Third, and most important he is allowed to retain some of the equity. This is the only possibility for the manager in order to transfer wealth across states. Other possibilities to transfer wealth are not available for the manager, that is, he is not allowed to take positions in other managers' projects.

It is important to notice that both agents determine the structure of both, the technology and the debt market. Without managers the access to the technology would be blocked. Since financing in this case is not necessary the debt returns are equal to zero. Without consumers a manager, because of his imposed poverty, does not receive financing in total and therefore is not able to undertake the project.

To formalize these thoughts let there be a finite number of managers, $k \in \{1, ..., K\} = \mathcal{K}$, in the economy. Let $x^k = (x_0^k, x_1^k, ..., x_S^k)$ and $w^k = (w_0^k, w_1^k, ..., w_S^k)$ denote the managers' consumption and endowment vector, respectively. Denote further the amount of investment by i_k^k, debt by d_k^k, and share of equity held by the manager by ζ_k^k. The first and second period budget constraint read:

$$x_0^k - w_0^k \leq -p_k i_k^k + q_k d_k^k + \left(1 - \zeta_k^k\right)\left(p_k i_k^k - q_k d_k^k\right), \tag{3.4}$$

$$x_s^k - w_s^k \leq \varrho_s^k i_k^k - R_s^k d_k^k - \left(1 - \zeta_k^k\right)\left(\varrho_s^k i_k^k - R_s^k d_k^k\right), \quad s = 1, ..., S. \tag{3.5}$$

It seems at first sight a little complicated to write the manager's budget constraint in such a way but it is more intuitive. Consider the following simple illustration: suppose there is a pie of total investments $p_k i_k^k$. In order for the manager to undertake the project he sells part of the pie, $q_k d_k^k$, as debt to lenders. The remaining pie, $p_k i_k^k - q_k d_k^k$, he sells as equity to consumers. $(1-\zeta_k^k)$ is exactly this share. The residual share the manager retains as his own share in the projects' equity. If $\zeta_k^k = 1$ the manager keeps the project for himself and does not sell any equity to consumers.

For later sections we write the above constraints in matrix notation. Define $i^k = (0,...,0,i_k^k,0,...0)'$ and $d^k = (0,...,0,d_k^k,0,...0)'$ as the manager's investment and debt vector, respectively. Let $(1-\zeta_k^k) = \big(0,...,0,(1-\zeta_k^k),0,...,0\big)$ be the manager's vector of consumer shares in the projects' equity. Since managers' are only intermixture of shares, amount and price in the equity claim we define the following vector operation:

Definition 3.1. Suppose two column vectors $x, y \in \mathbb{R}^K$. Let the componentwise vector multiplication, $\odot$, be defined as:

$$x \odot y = (x_1 y_1, ..., x_K y_K)'. \tag{3.6}$$

We will transform the manager's budget equations step-by-step in order to apply and explain the above definition. Using the technology and debt market given in (3.1) and (3.2). The budget constraint simplifies to

$$x^k - w^k \leq \mathcal{P} i^k - \mathcal{D} d^k + \mathcal{D}\left[(1-\zeta^k) \odot d^k\right] - \mathcal{P}\left[(1-\zeta^k)\odot i^k\right],$$

with $(1-\zeta^k)\odot d^k = \big(0,...,0,(1-\zeta_k^k)d_k^k,0,...,0\big)' \in \mathbb{R}_+^K$, $(1-\zeta_{k'}^k)d_{k'}^k = 0$, for all $k' \neq k$.

$$\begin{aligned} x^k - w^k &\leq \mathcal{P}\left[i^k - (1-\zeta^k)\odot i^k\right] - \mathcal{D}\left[d^k - (1-\zeta^k)\odot d^k\right] \\ x^k - w^k &\leq \mathcal{P}\left[\zeta^k \odot i^k\right] - \mathcal{D}\left[\zeta^k \odot d^k\right] \end{aligned} \tag{3.7}$$

which is equal to

$$\begin{pmatrix} -p_1 & \cdots & -p_K \\ \varrho_1^1 & \cdots & \varrho_1^K \\ \vdots & \ddots & \vdots \\ \varrho_S^1 & \cdots & \varrho_S^K \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \zeta_k^k i_k^k \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \begin{pmatrix} -q_1 & \cdots & -q_K \\ R_1^1 & \cdots & R_1^K \\ \vdots & \ddots & \vdots \\ R_S^1 & \cdots & R_S^K \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \zeta_k^k d_k^k \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.8}$$

Using the whole market $\mathcal{M}$ the manager's buget constraint simplifies to

$$x^k - w^k \leq \mathcal{M} \begin{bmatrix} \zeta^k \odot i^k \\ -\zeta^k \odot d^k \end{bmatrix}. \tag{3.9}$$

It is clear that because of the multiplicative structure of the above portfolio, formed by the manager, that wealth is only transferred if and only if the manager undertakes his investment project.

The above transformation shows that the manager, although trading exclusively on the technology market and debt market, is able to generate an additional asset class, the equity. He forms a portfolio of the underlying (investment) and debt or bond. In order to receive the equities' payoff the manager is obliged to exercise the call option on the project. The strike is given by the total amount of debt which has to be served first. If not the project is lost to the debt holders and the payoff to equity holders is zero.

The columns of the market, $\mathcal{M}$, generate a subspace in the commodity space $\mathbb{R}^{S+1}$, which we will denote by $\langle\mathcal{M}\rangle$. The subspace, called market subspace, summarizes the opportunities of wealth transfers offered by the market. Let τ denote a vector of wealth transfers, then $\langle\mathcal{M}\rangle = \{\tau \in \mathbb{R}^{S+1} \mid \tau = \mathcal{M}z, \quad z \in \mathbb{R}^{2K}\}$. Notice that because of the implied market the market subspace varies according to the particular financing chosen by the manager. This is due to the dependence of the default mechanism on the level of debt and investment chosen by the manager.

Each manager has a preference ordering defined on the consumption set $\mathbb{R}^{S+1}_{+}$ which is complete, transitive and continuous.[2] As Mas-Colell, Whinston, and Green (1995, p. 47) show (and many others) there exists a continuous utility function representing the consumers preference relation. For the utility function we impose the following assumption:

Assumption 1 (Strong monotonicity)
(i) $U^k : \mathbb{R}^{S+1}_{+} \to \mathbb{R}$ be a continuous utility function on $\mathbb{R}^{S+1}_{+}$.
(ii) For any $x, y \in \mathbb{R}^{S+1}_{+}$ with $x \geq y$ and $x \neq y$, $U^k(x) > U^k(y)$.[3]

[2] The preference relation $\succeq$ is complete, if $\forall x, y \in \mathbb{R}^{S+1}_{+}$, $x \succeq y$ or $y \succeq x$ or both. It is transitive, if $\forall x, y, z \in \mathbb{R}^{S+1}$ and $x \succeq y$, $y \succeq z$ then $x \succeq z$.

[3] See e.g. Mas-Colell, Whinston, and Green (1995, p. 42) for details and implications.

The manager solves the following optimization problem:

$$\max_{\substack{(x^k,i_k^k,d_k^k,\zeta_k^k)\\ \in\mathbb{R}_+^{S+1}\times\mathbb{R}_+\times\mathbb{R}_+\times[0,1]}} \left\{ U^k(x^k) \;\middle|\; x^k - w^k \leq \mathcal{M}\left[\begin{array}{c} \zeta^k \odot i^k \\ -\zeta^k \odot d^k \end{array}\right] \right\}. \tag{3.10}$$

3.4 Consumers

Let there be $i \in \{1, ..., I\} = \mathcal{I}$ consumers in the economy. Each consumer chooses a vector of consumption $x^i = (x_0^i, x_1^i, ..., x_S^i)'$ and is given an exogenous endowment vector $w^i = (w_0^i, w_1^i, ..., w_S^i)'$. In order to transfer wealth across states the consumers are allowed to take positions in the managers' equity and debt. For this purpose let $b^i = (b_1^i, ..., b_K^i)'$ and $\theta^i = (\theta_1^i, ..., \theta_K^i)'$ denote the consumers' portfolio for debt and equity respectively. The first and second period budget constraints, omitting unnecessary subscripts, read

$$x_0^i - w_0^i \leq \sum_k q_k b_k^i - \sum_k \theta_k^i \left(p_k i^k - q_k d^k\right) \tag{3.11}$$

$$x_s^i - w_s^i \leq -\sum_k R_s^k b_k^i + \sum_k \theta_k^i \left(\varrho_s^k i_k^k - R_s^k d_k^k\right) \quad , s = 1, .., S \tag{3.12}$$

with $b_k^i \leq 0$ and θ_k^i taking values in $[0, 1]$. Hence, we allow only for long positions in both assets and no short sales. The budget equations simplify to

$$x^i - w^i \leq \mathcal{P}\left[\theta^i \odot \sum_k i^k\right] - \mathcal{D}\left[b^i + \theta^i \odot \sum_k d^k\right] \tag{3.13}$$

with $\theta^i \odot \sum_k i^k = \left(\theta_1^i i_1^1, ..., \theta_K^i i_K^K\right)'$ and $b^i + \theta^i \odot \sum_k d^k = \left(b_1^i + \theta_1^i d_1^1, ..., b_K^i + \theta_K^i d_K^K\right)'$. Using the total market, $\mathcal{M}$, the budget equation reads:

$$x^i - w^i \leq \mathcal{M}\left[\begin{array}{c} \theta^i \odot \sum_k i^k \\ -b^i - \theta^i \odot \sum_k d^k \end{array}\right]. \tag{3.14}$$

Notice that the consumers' portfolio vector (obviously) depends on the managers' decisions on the amount of debt and investment. The consumers' total investment in the technology is linear in the managers' amount invested in the technology.

Assumption 1 is also valid for the consumers. They solve the following optimization problem:

$$\max_{\substack{(x^i,b^i,\theta^i)\\ \in\mathbb{R}_+^{S+1}\times\mathbb{R}_-^K\times[0,1]^K}} \left\{ U^i(x^i) \;\middle|\; x^i - w^i \leq \mathcal{M}\left[\begin{array}{c} \theta^i \odot \sum_k i^k \\ -b^i - \theta^i \odot \sum_k d^k \end{array}\right] \right\}. \tag{3.15}$$

4 Equilibrium

We first show the market clearing conditions in the following proposition and subsequently define the equilibrium for the proposed economy. Non-trivial issues concerning equilibrium will follow.

Proposition 4.1. *Given Assumption 1 the market clearing conditions are given by:*

$$\zeta_k^k + \sum_i \theta_k^i = 1 \quad \textit{and} \quad \sum_i b_k^i + d_k^k = 0, \quad \forall\ k = 1, ..., K. \tag{4.1}$$

Remark 4.2. Notice that the artificial equity market has to clear in order for the commodity market to clear. In the proof it will be shown that equity market clearance is inevitable for the clearance of the debt market. Recall that debt positions taken by consumers are negative since they are considered lenders, i.e., $b_k^i \leq 0$ for all k.

PROOF. First we need to show that

$$\sum_k \left(\zeta^k \odot i^k\right) + \sum_i \left(\theta^i \odot \sum_k i^k\right) = \sum_k i^k$$

which is equal to

$$\begin{pmatrix} i_1^1 \\ \vdots \\ i_K^K \end{pmatrix} \left[\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} - \begin{pmatrix} \sum_i \theta_1^i \\ \vdots \\ \sum_i \theta_K^i \end{pmatrix} - \begin{pmatrix} \zeta_1^1 \\ \vdots \\ \zeta_K^K \end{pmatrix} \right] = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{4.2}$$

The first clearing condition is immediate. Secondly we need to show that

$$\sum \left(\zeta^k \odot d^k\right) + \sum_i \left(b^i + \theta^i \odot \sum_k d^k\right) = 0 \tag{4.3}$$

which is equal to

$$\begin{pmatrix} d_1^1 \left(\sum_i \theta_1^i + \zeta_1^1\right) \\ \vdots \\ d_K^K \left(\sum_i \theta_K^i + \zeta_K^K\right) \end{pmatrix} = \begin{pmatrix} -\sum_i b_1^i \\ \vdots \\ -\sum_i b_K^i \end{pmatrix}. \tag{4.4}$$

□

Remark 4.3. It is essential in order to achieve clearance in the debt market that the equity market clears.

For an equilibrium to exist we need the set optimized decision variables, resulting from the managers' and consumers' optimization problem, and a set of investment and debt prices such that markets clear. For the proposed economy we define equilibrium as follows:

Definition 4.4. An equilibrium for the above economy is a pair $\left(\left(\bar{x}, \bar{\imath}, \bar{d}, \bar{\zeta}, \bar{b}, \bar{\theta}\right), (\bar{p}, \bar{q})\right) \in \mathbb{R}_{+}^{(S+1)(I+K)} \times \mathbb{R}_{+}^{K} \times \mathbb{R}_{+}^{K} \times [0,1]^{K} \times \mathbb{R}_{-}^{IK} \times [0,1]^{I} \times \mathbb{R}_{+}^{K} \times \mathbb{R}_{+}^{K}$, such that

(i) The manager solves his optimization problem given in (3.10).

(ii) The consumer solves his optimization problem given in (3.15).

(iii) The market clearing conditions in (4.1) hold.

The issues about sensitivity on the existence of equilibrium are manifold. Looking closer at the managers' budget constraint one may observe that we have been very demanding in that the only incentive to transfer wealth across states is by undertaking the project with the given technology. However, the decision to undertake the project is not entirely up to the manager since part of the financing is provided by the consumers. The model is very sensitive in the sense that, once financing has been approved, the manager might sell only debt and retains all the equity. This is due to the fact that the aggregate supply and/or aggregate demand function are not continuous and therefore exhibit jumps. In worst case no equilibrium exists because both function never intersect due to discontinuities.

5 Open Issues and Outlook

The above economy is, for now, only proposed and might not exist in a general equilibrium sense. The issues concerning equilibrium is the crucial point. In order to clarify on the issues of equilibrium the next step is to numerically find an non-trivial equilibrium. Non-trivial meaning that both type of agents in the model share equity in the investment project. This task is performed by a simulation study solving the model backwards. Given some investment and debt prices the agents solve their optimization problem. If utility attains a maximum and if the market clearing conditions states in (4.1) hold an equilibrium is found. Unfortunately the problem to solve is not linear, such that, if excess demand for the debt is positive raising debt prices is not sufficient in order to clear the market. Since equity is considered, changing debt prices has also an immediate effect on the equity and therefore the portfolio choice.

An interesting topic is also to have a closer look at the endogenous asset structure and the according market subspace. The consumers in the present model use the managers to transfer wealth across states. However, with every manager not getting the proper finance and therefore not undertaking the project the consumers lose degrees of freedom in how to distribute their wealth properly. Manager therefore offer a so called spanning service for the consumers. The span is trimmed in case some managers do not undertake their project. However, there may exist consumers who have an interest in expanding the span in order to transfer wealth properly. If equilibrium is established it would be interesting to see whether we can impose conditions such that managers receive financing although they would not undertake the project in case of omitting these conditions.

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